Stokes law in complex liquids

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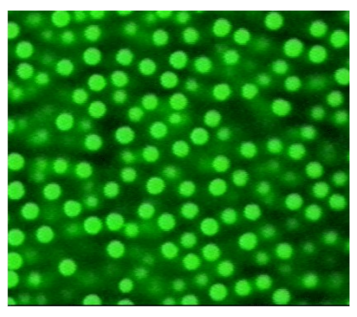
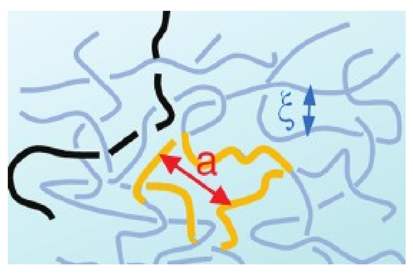
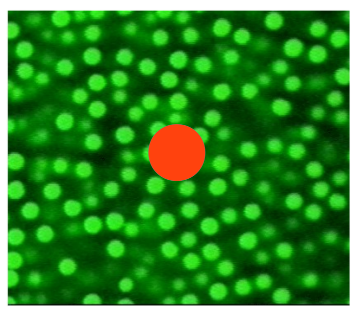
Motivation and goals

Example of complex liquids: suspensions, polymer solutions





What is the form of the Stokes law in complex liquid, when it is described by Stokes equations with wave vector dependent viscosity?



Drag force on spherical particle moving in complex liquid:

$$\mathbf{F} = \zeta(a)\,\mathbf{U}$$

Friction coefficient

Stokes law in simple liquids:

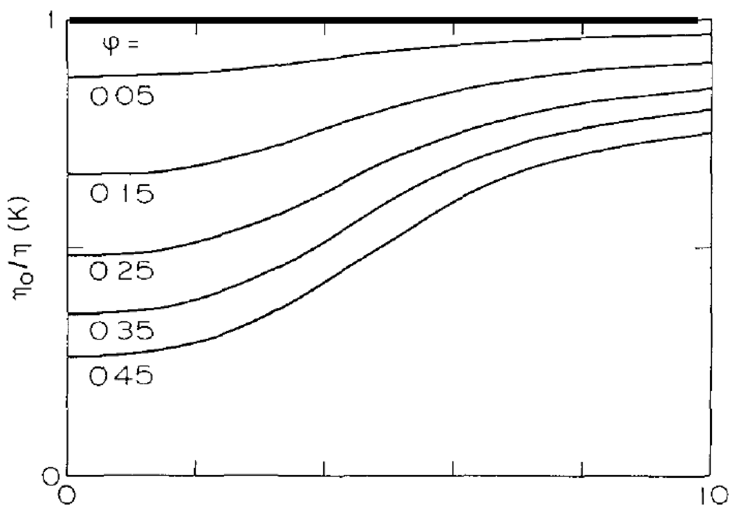
$$\zeta(a) = 6\pi\eta a$$

Simple and complex liquids

simple liquid: $\eta = \mathrm{const}$ → complex liquid: $\eta(k)$

Second order partial diff. Eqs. → Infinite order partial diff. Eqs.

$\eta\triangle$ → $\eta(\triangle)\triangle$

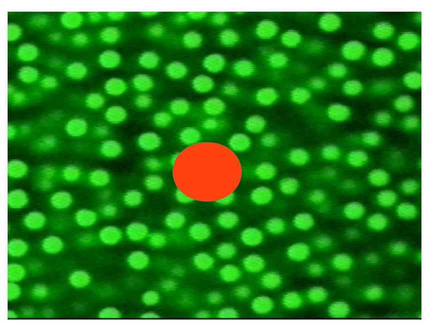


Beenakker, Physica A, 128, 48 (1984)

Formulation of the problem

Stokes equations in complex liquids with wave vector dependent viscosity:

$$\begin{aligned} i\mathbf{k}p + k^2\eta(k)\,\hat{\mathbf{v}}(\mathbf{k}) &= 0,\\ \mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{k}) &= 0. \end{aligned}$$



Boundary conditions:

$\mathbf{v}(\mathbf{r}) = \mathbf{U}$ for $|r| = a$

$\mathbf{v}(\mathbf{r}) \to 0$ for $r \to \infty$

$$\mathbf{F} = \zeta(a)\,\mathbf{U}$$

What is the friction coefficient $\zeta(a)$?

Linearity and spherical symmetry (isotropic fluid, spherical particle) strongly simplifies derivation in simple fluids, where $\eta = \mathrm{const}$

Idea of the generalization of the Stokes law to the case of complex liquids when described by Stokes equations with wave vector dependent viscosity:

- derive the Stokes law in simple liquids in Fourier space and with use of spherical symmetry
- generalize the above derivation to the case of the wave-vector dependent viscosity

Stokes law – generalization to the case of complex liquids

Ansatz for velocity field:

$$\hat{\mathbf{v}}(\mathbf{k}) = \hat{\mathbf{v}}_0(\mathbf{k}) + c\,\hat{\mathbf{v}}_1(\mathbf{k})$$

$$\begin{aligned} i\mathbf{k}p_0 + k^2\eta(k)\,\hat{\mathbf{v}}_0(\mathbf{k}) &= \mathbf{F},\\ \mathbf{k}\cdot\hat{\mathbf{v}}_0(\mathbf{k}) &= 0. \end{aligned} \qquad \hat{\mathbf{v}}_0(\mathbf{k}) = \frac{1}{\eta(k)\,k^2}\left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F}$$

$$\hat{\mathbf{v}}_1(\mathbf{k}) = \eta(k)\,k^2\,\hat{\mathbf{v}}_0(\mathbf{k}) = \left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F} \qquad \mathbf{v}_1(\mathbf{r}) = \frac{1}{4\pi r^3}\left(\mathbf{1} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\mathbf{F}$$

Boundary conditions on the surface of particle, applied to the above ansatz lead to

$$\mathbf{v}(a\hat{\mathbf{r}}) = \mathbf{U}$$

$$\mathbf{U} = \frac{1}{(2\pi)^3}\int d^3k\; e^{ia\hat{\mathbf{r}}\mathbf{k}}\,\frac{1}{\eta(k)\,k^2}\left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F} + c\,\frac{1}{4\pi r^3}\left(\mathbf{1} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\mathbf{F}$$

which further yields

$$\zeta(a) = 3\pi^2 \Big/ \left(\int_0^\infty dk\; j_0(ka)/\eta(k)\right)$$

$$j_0(x) = \sin(x)/x$$

Fourier transform

scale dependent viscosity from friction coefficient:

$$\eta(k) = \frac{1}{6\pi k^2}\left[\int_0^\infty da\; a^2\,\frac{j_0(ak)}{\zeta(a)}\right]^{-1}$$

Application – from diffusion to scale dependent viscosity

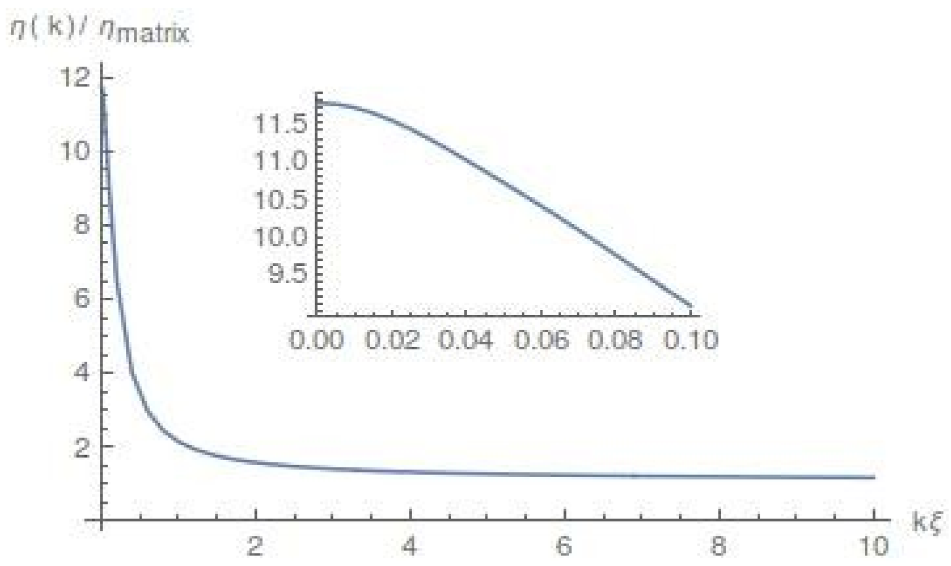
Under very simplifying assumptions, the friction coefficient may be inferred from measurements of diffusion coefficient of probe particles in cytoplasm of living cells [1]. Phenomenological approach leads to the following expression for the friction coefficient

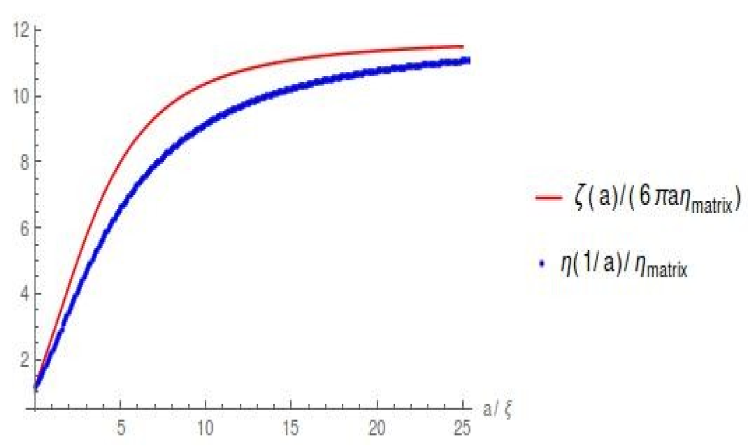
$$\zeta(a) = 6\pi\eta_{\mathrm{matrix}}\,a\,\exp\left[\left(\frac{R_{\mathrm{eff}}}{\xi}\right)^{\alpha}\right] \qquad R_{\mathrm{eff}}^{-2} = R_h^{-2} + a^{-2}$$

with the following coefficients for Swiss 3T3 mammalian cells

$$\xi = 7nm, \quad R_h = 30nm, \quad \alpha = 0.62$$

Basing on our formula (blue frame) we can infer the form of scale dependent viscosity from measurements of diffusion coefficient of probe particles in complex liquid:





[1] Kalwarczyk et al., *Nano Lett.* 2011, 11, 2157–2163