Short-time transport properties of suspensions of spherical particles

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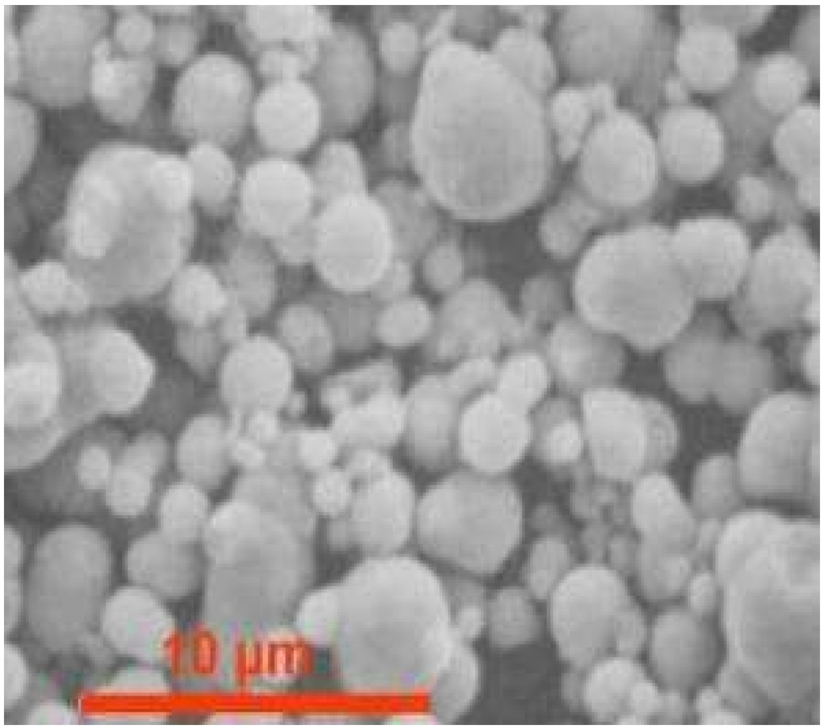
Introduction - suspensions

minute particles in liquid



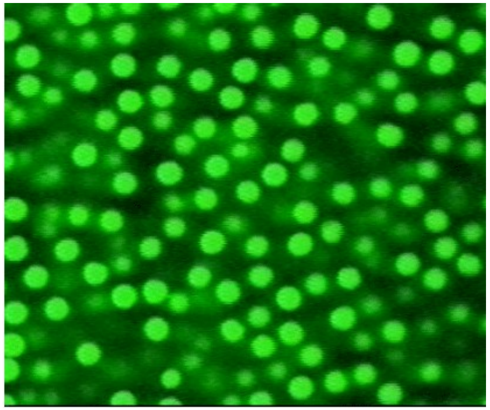
Liquid:
-temperature T
-viscosity μ
-density of the fluid ρ_f

Particles:
-radius a
-density of material ρ_p
-volume fraction ϕ

milk, blood,...

Goal of our work

Monodisperse suspension
of spherical particles

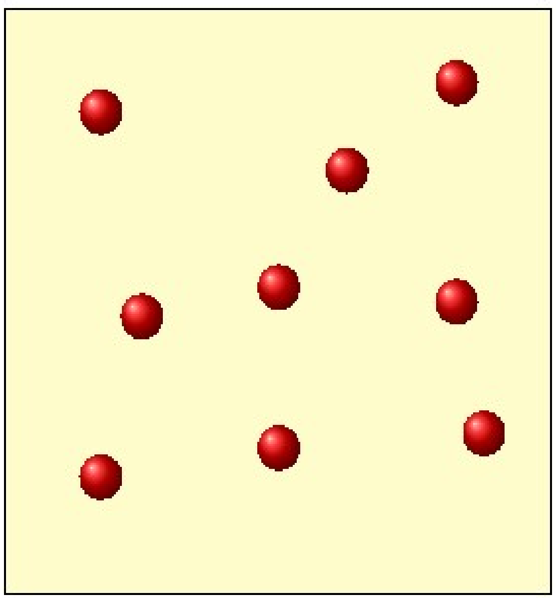


Transport properties (short time):
-effective viscosity
-sedimentation coefficient
-diffusion coefficient

Over 100 years of research - still an open question

Hard-sphere suspension

Unbounded liquid,
N particles in configuration $X \equiv \mathbf{R}_1, \ldots, \mathbf{R}_N$



Stokes equations:

$$\begin{aligned} \nabla p\left(\mathbf{r}\right) - \eta \Delta \mathbf{v}\left(\mathbf{r}\right) &= 0 \\ \nabla \cdot \mathbf{v}\left(\mathbf{r}\right) &= 0 \end{aligned}$$

Stick boundary conditions,

$\mathbf{v}(\mathbf{r}) \rightarrow \mathbf{v}_0(\mathbf{r})$ *for* $r \rightarrow \infty$

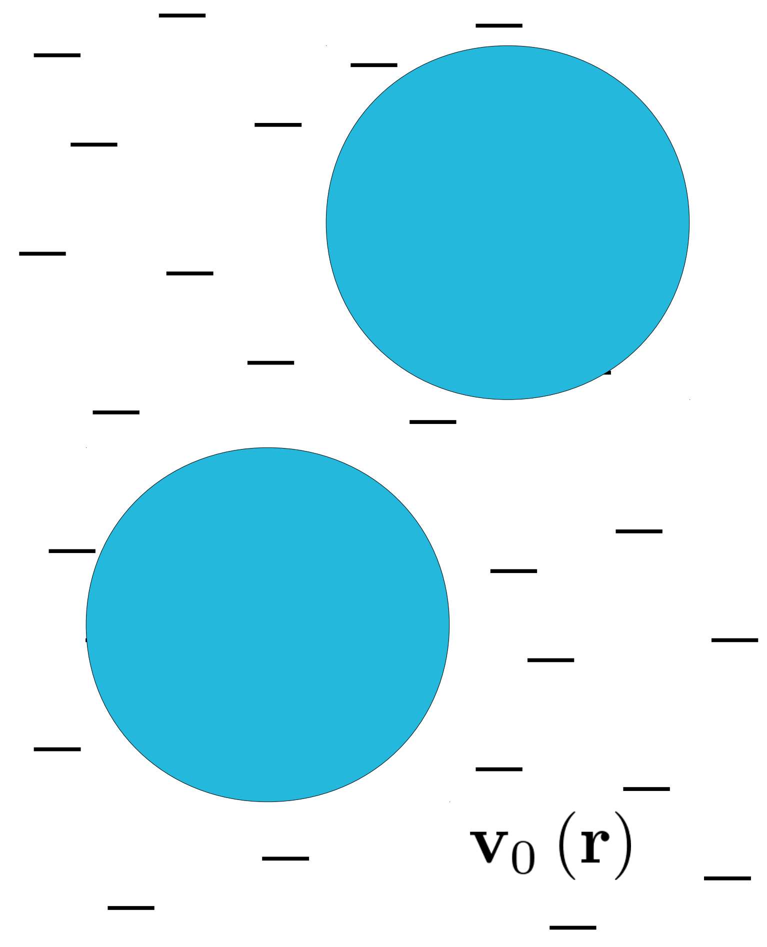
Effective viscosity

Landau: effective viscosity related to force on the surface of particles

$$\mathbf{f}_i(\mathbf{r}) = -\sigma(\mathbf{r}; X)\hat{\mathbf{n}}(\mathbf{r})\delta(|\mathbf{r} - \mathbf{R}_i| - a)$$

stress tensor

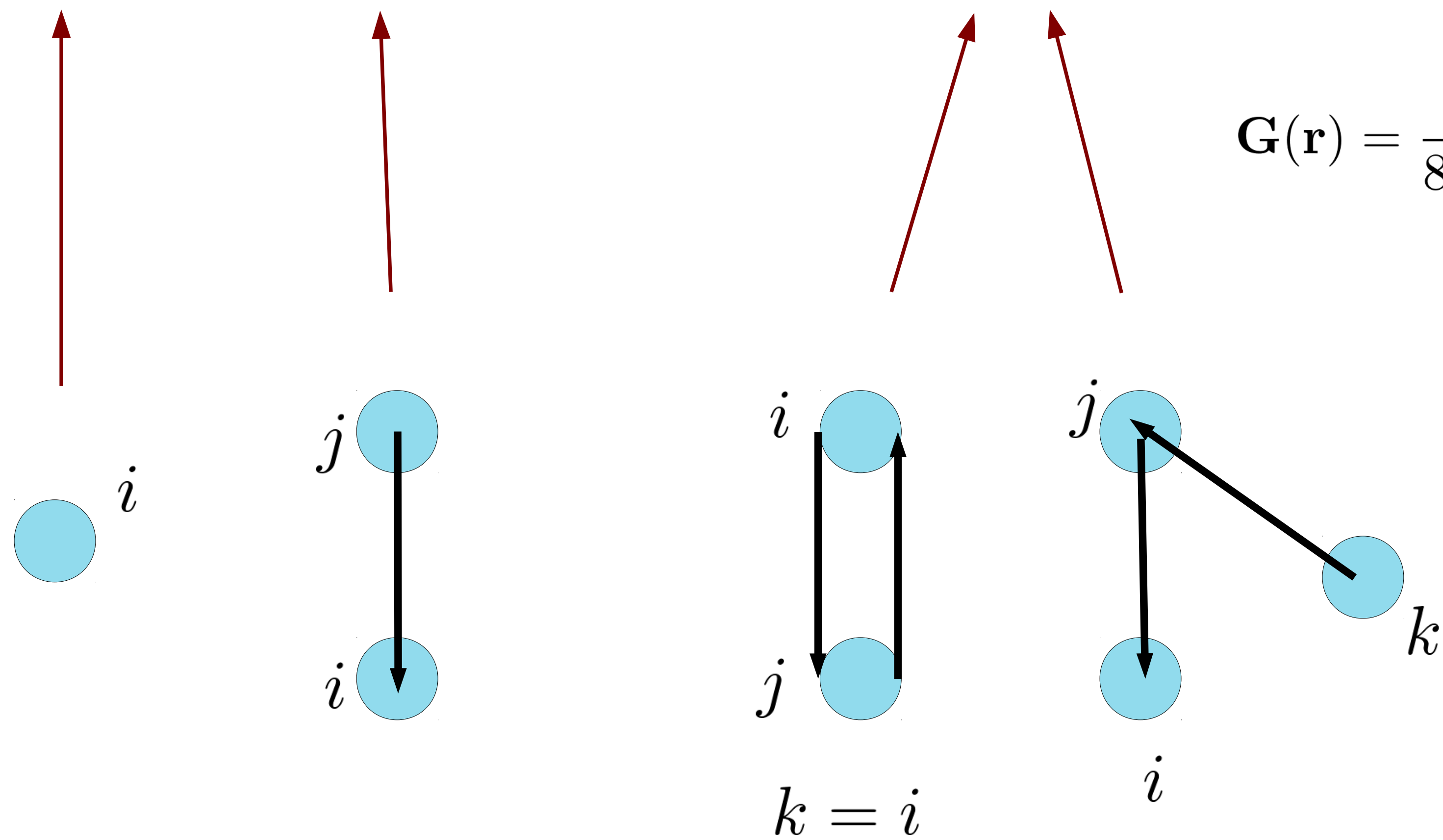
vector normal to the surface of particle i



Scattering series

$$\mathbf{f}_i = \left(\mathbf{M}(i) + \sum_{j \neq i} \mathbf{M}(i)\mathbf{G}\mathbf{M}(j) + \sum_{j \neq i} \sum_{k \neq j} \mathbf{M}(i)\mathbf{G}\mathbf{M}(j)\mathbf{G}\mathbf{M}(k) + \ldots \right) \mathbf{v}_0$$

$$\mathbf{G}(\mathbf{r}) = \frac{1}{8\pi\eta} \frac{\mathbf{1} + \hat{\mathbf{r}}\hat{\mathbf{r}}}{r}$$



suspension <=> dielectrics <=> other systems

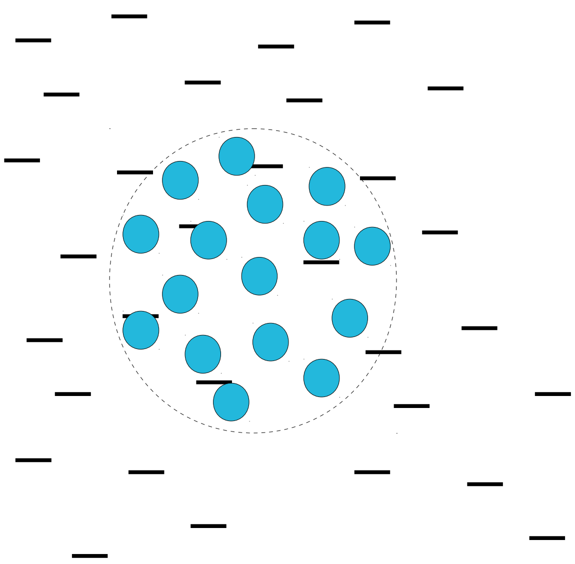
Transport properties – history and scattering series

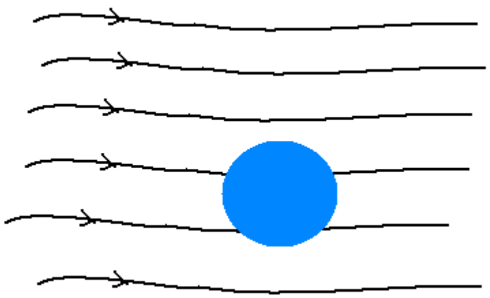
Einstein 1905 (corrected):

$$\eta_{eff} = \eta(1 + \frac{5}{2}\phi)$$

$$\phi = \frac{4}{3}\pi a^3 n$$

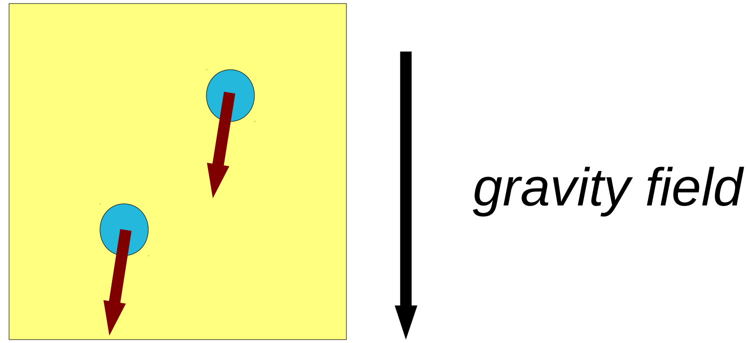
Single particle in ambient (shear) flow $\mathbf{v}_0(\mathbf{r})$

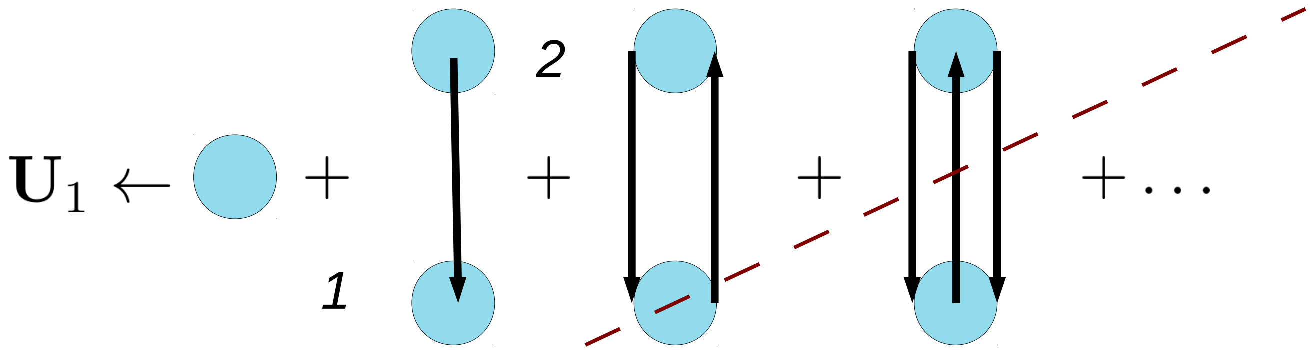


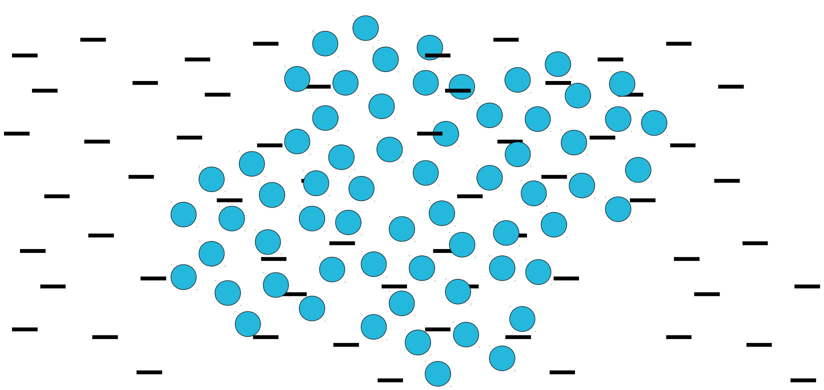


- *Finite system*
- *Hydrodynamic interactions neglected (no reflections, single particle)*

Hydrodynamic interactions – Smoluchowski (1911)







$$\int d^3r \left|\mathbf{G}(\mathbf{r})\right| = \infty$$

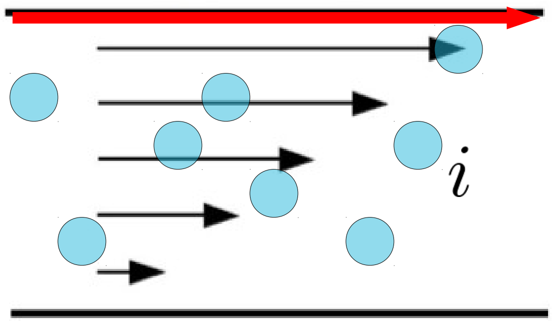
$$\mathbf{G}(\mathbf{r}) = \frac{1}{8\pi\eta}\frac{\mathbf{1}+\hat{\mathbf{r}}\hat{\mathbf{r}}}{r}$$

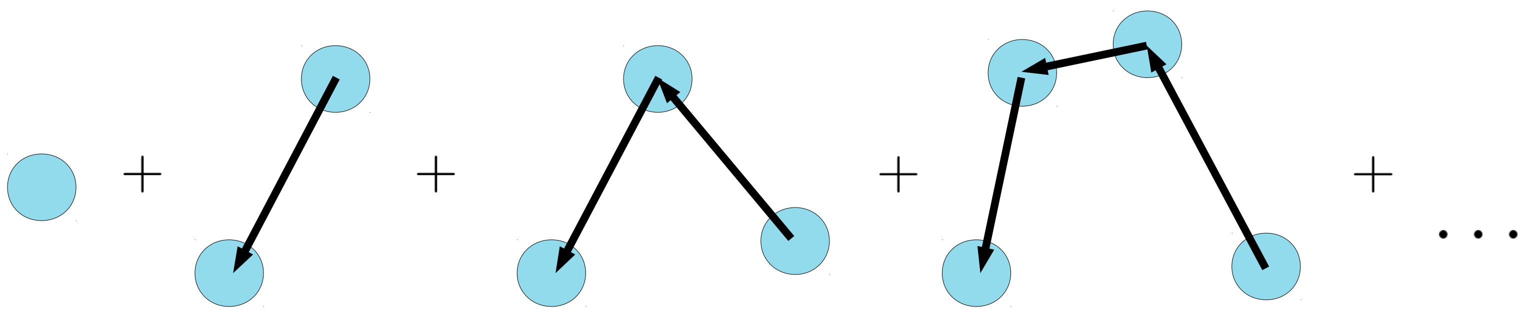
Well defined expression for effective viscosity?

Beyond diluted suspensions

Saito (1950):

-extension of Einstein work on mean-field level





$$\mathbf{M}(\mathbf{R}_i)\mathbf{G}\mathbf{M}(\mathbf{R}_j) \rightarrow W(\mathbf{R}_i - \mathbf{R}_j)\mathbf{M}(\mathbf{R}_i)\mathbf{G}\mathbf{M}(\mathbf{R}_j)$$

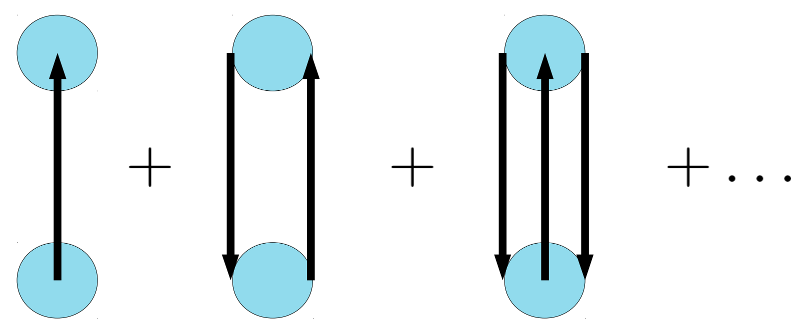
vanishes when two particles overlap

$$\frac{\eta_{eff}}{\eta} = \frac{1 + \frac{3}{2}\phi}{1 - \phi}$$

non-absolutely convergent integrals!

Two-particle hydrodynamic interactions (1963)

$$\frac{\eta_{eff}}{\eta} = 1 + \frac{5}{2}\phi + a_2\phi^2 + \ldots$$



absolute convergence

$$\int d^3r \left|\mathbf{G}(\mathbf{r})\right| = \infty$$

(well defined expression on two-body level)

Peterson, Fixman (1963): $a_2 \approx 4.31784$

Batchelor, Green (1972): $a_2 \approx 5.2$

(ad hoc renormalization)

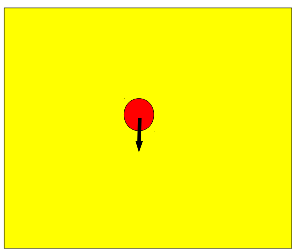
Problem with long-range HI still not solved

Hydrodynamic interactions

Many-body character

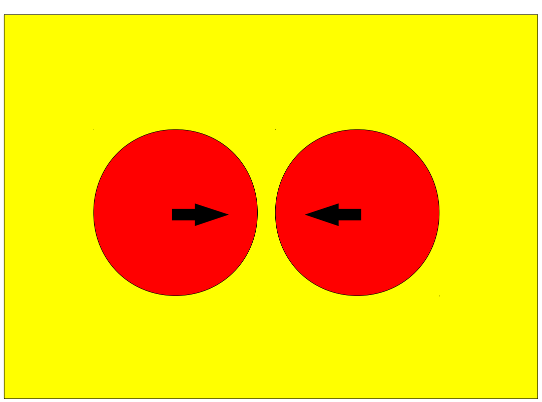
two-body approximation relevant for volume fractions less than about 5%

Long-range character



$$\mathbf{v}\left(\mathbf{r}\right) \sim \mathbf{G}\left(\mathbf{r}\right) \cdot \mathbf{F}$$

Strong interactions of close particles



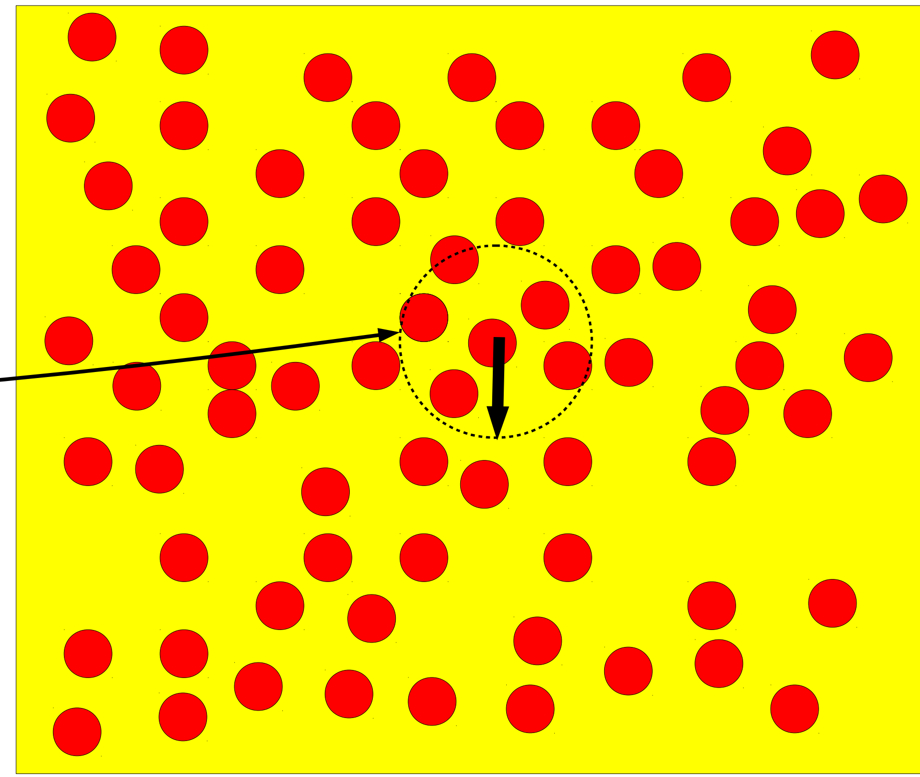
For constant velocities
asymptotically infinite drag force
(Jeffrey, Onishi (1984))

Effective Green function
– includes all three features of hydrodynamic interactions

Flow caused by force acting on particles in the area

total force acting on particles in the area

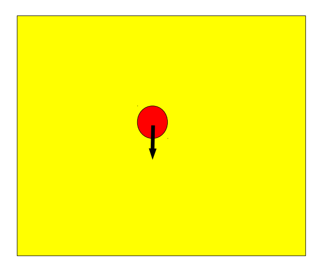
$$\mathbf{v}\left(\mathbf{r}\right) \sim \mathbf{G}_{\mathrm{eff}}\left(\mathbf{r}\right)\mathbf{F}$$



effective Green function (effective propagator):

$$\mathbf{G}_{\mathrm{eff}}(\mathbf{r}) \sim \frac{1}{8\pi\eta_{\mathrm{eff}}}\frac{\mathbf{1}+\hat{\mathbf{r}}\hat{\mathbf{r}}}{r} = \frac{\eta}{\eta_{\mathrm{eff}}}\mathbf{G}(\mathbf{r})$$

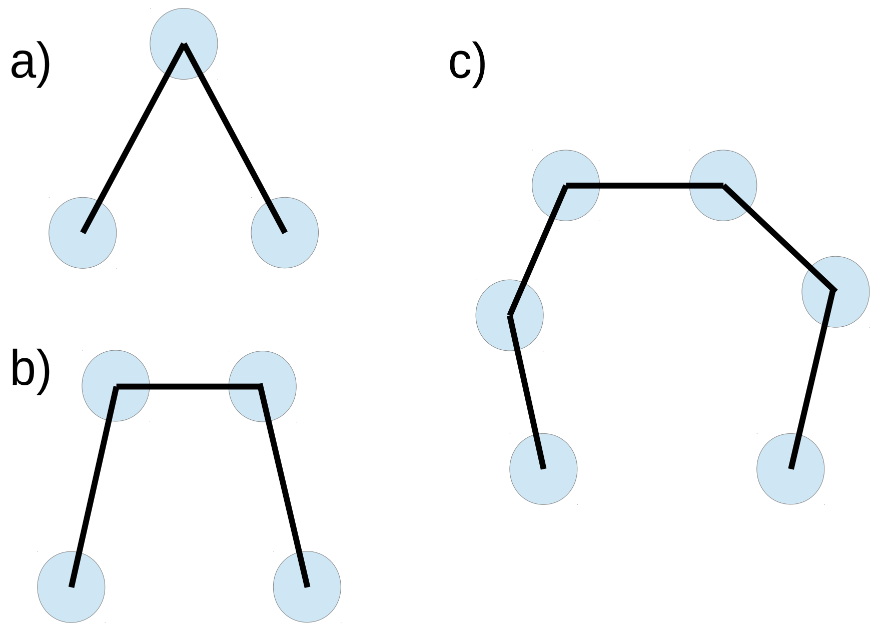
at the distance

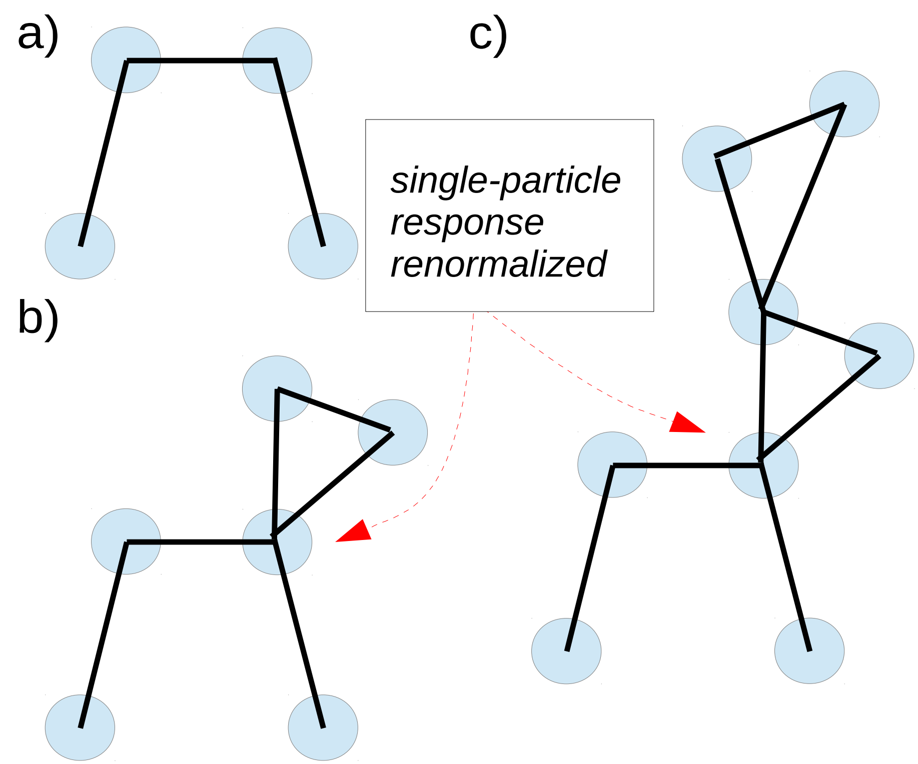


$$\mathbf{v}\left(\mathbf{r}\right) \sim \mathbf{G}\left(\mathbf{r}\right)\cdot\mathbf{F}$$

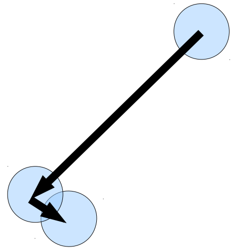
Beenakker-Mazur method (1983)

Idea of the method – resummation of certain class of hydrodynamic interactions – 'ring-selfcorrelations'





No correlations in position between particles in the above resummed terms



Beenakker and Mazur scheme

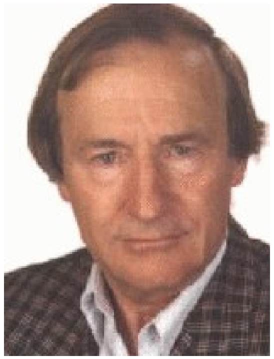
Beenakker and Mazur scheme – expansion in density fluctuations (1983). The most comprehensive statistical physics theory for short times properties of suspension nowadays

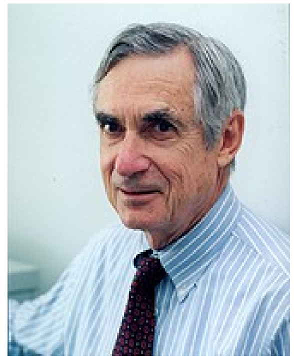
- ✓ *Many-body character*
- ✓ *Long-range character*
- ✗ *Strong interactions of close particles*

No satisfactory statistical physics method including the above three features

Lubrication important!

1982 – problem of long-range HI solved





B. U. Felderhof,[1] G. W. Ford,[2] and E. G. D. Cohen[3]

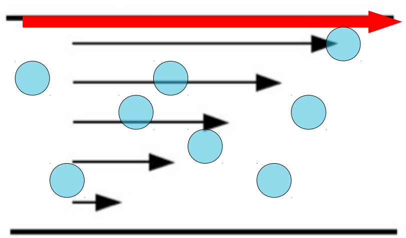
Received August 24, 1981

We derive a cluster expansion for the electric susceptibility kernel of a dielectric suspension of spherically symmetric inclusions in a uniform background. This also leads to a cluster expansion for the effective dielectric constant. It is shown that the cluster integrals of any order are absolutely convergent, so that the dielectric constant is well defined and independent of the shape of the sample in the limit of a large system. We compare with virial expansions derived earlier in

dielectric <=> suspension

Response of suspension (effective viscosity)

Viscosity by relation between pressure tensor and average flow of suspension (Landau):



average velocity field of suspension

$$\langle \mathbf{f}(\mathbf{R}) \rangle = \int d^3 r' \mathbf{T}^{irr}(\mathbf{R}, \mathbf{r}') \langle \mathbf{v}(\mathbf{r}') \rangle$$

average surface force (dipole)

viscosity operator

Effective viscosity coefficient is given directly by the response operator T^{irr}

Felderhof, Ford, Cohen – cluster expansion (1982)

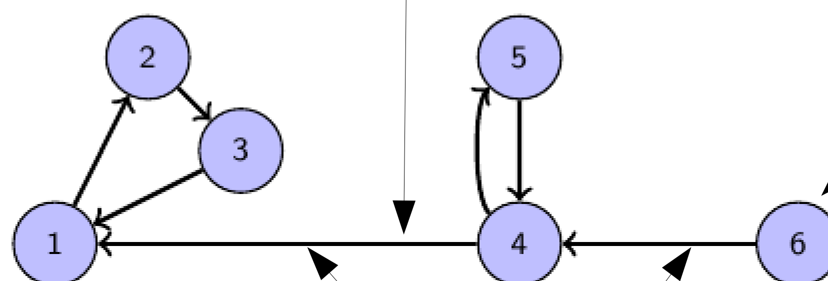
$$T^{irr} = \sum_{g=1}^{\infty} \sum_{C_1 \ldots C_g} \int dC_1 \ldots dC_g b(C_1|\ldots|C_g) S_I(C_1) G \ldots G S_I(C_g)$$

block distribution function (configurations of particles)

Oseen tensor: $G = \frac{1}{8\pi\eta} \frac{\mathbf{1} + \hat{\mathbf{r}}\hat{\mathbf{r}}}{r}$

Example of scattering sequence (many-body):

$$M(1)GM(3)GM(2)GM(1) \times$$
$$G \times M(4)GM(5)GM(4) \times G \times M(6)$$



$M(6)$

short range hydrodynamic interactions (strong interactions of close particles)

long range hydrodynamic interactions

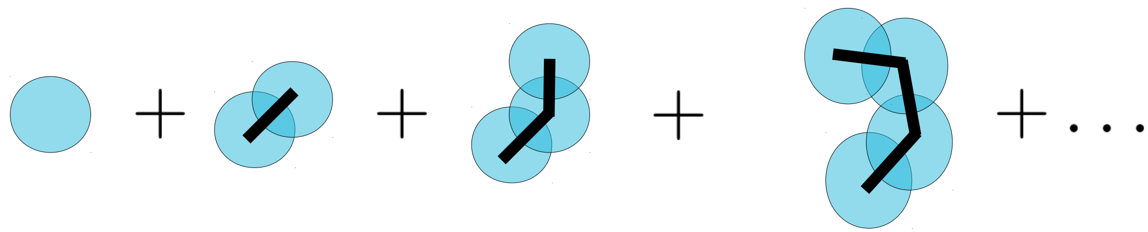
$$S_I(123)GS_I(45)GS_I(6)$$

$$C_1 \equiv 123 \quad C_2 \equiv 45 \quad C_3 \equiv 6$$

Problem with long-range HI solved

Felderhof, Ford, Cohen – microscopic explanation of Clausius-Mossotti (Saito) formula (1983)

$$\mathbf{T}^{irr} \leftarrow$$



$$\langle \mathbf{f}(\mathbf{R}) \rangle = \int d^3 r' \mathbf{T}^{irr}(\mathbf{R}, \mathbf{r}') \langle \mathbf{v}(\mathbf{r}') \rangle$$

The following definition

$$\mathbf{T}^{irr}_{CM} = \mathbf{T}^{irr} \left(1 + [h\mathbf{G}]\, \mathbf{T}^{irr} \right)^{-1}$$

and approximate closure relation

$$\mathbf{T}^{irr}_{CM} \approx n_1 \hat{\mathbf{M}}$$

lead to Saito formula

$$\frac{\eta_{eff}}{\eta} = \frac{1 + \frac{3}{2}\phi}{1 - \phi}$$

Our approach – renormalization of the propagator

Cluster expansion (1982):

short-range hydrodynamic interaction

$$T^{irr} = \sum_{g=1}^{\infty} \sum_{C_1 \ldots C_g} \int dC_1 \ldots dC_g b(C_1|\ldots|C_g) S_I(C_1) G \ldots G S_I(C_g)$$

block distribution function (configurations of particles)

Oseen tensor (pure liquid):

$$G = \frac{1}{8\pi\eta} \frac{\mathbf{1} + \hat{\mathbf{r}}\hat{\mathbf{r}}}{r}$$

Ring expansion (2015):

$$T^{irr} = \sum_{g=1}^{\infty} \sum_{C_1 \ldots C_g} \int dC_1 \ldots dC_g H(C_1|\ldots|C_g) S_I(C_1) G_{\mathrm{eff}} \ldots G_{\mathrm{eff}} S_I(C_g)$$

block correlation function
(configurations of particles);
H=b for g=1,2,
H different from b for g>2.

Effective Green function:

$$\mathbf{G}_{\mathrm{eff}}(\mathbf{r}) \sim \frac{\eta}{\eta_{\mathrm{eff}}} \mathbf{G}(\mathbf{r})$$

convergence

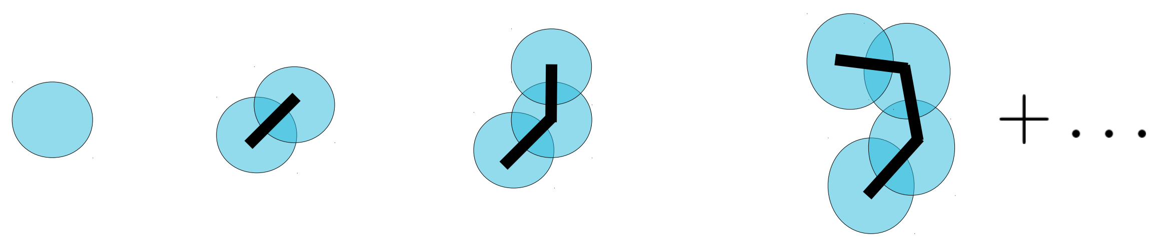
Generalization of Clausius-Mossotti approximation

Constructing approximate method by carrying over approximation from cluster expansion to ring expansion with the following modification:

$$G \Longrightarrow G_{\mathrm{eff}}$$

Clausius-Mossotti (Saito) approximation $\Longrightarrow$ *Generalized Clausius-Mossotti approximation*

(two-body hydrodynamic interactions incomplete – the same as in Beenakker and Mazur scheme)



Volume fraction, two-body correlation function $\Longrightarrow$ *transport properties*

Felderhof, Ford and Cohen cluster expansion (1982)

(effective viscosity)

average velocity field of suspension

$$\langle f \rangle (\mathbf{R}) = \int d\mathbf{r}' T^{irr}(\mathbf{R}, \mathbf{R}') \langle v \rangle (\mathbf{R}')$$

average surface force

$$\langle v(\mathbf{R}) \rangle = v_0(\mathbf{R}) + \int d\mathbf{r}' G(\mathbf{R}, \mathbf{R}') \langle f(\mathbf{R}') \rangle$$

$$\langle f(\mathbf{R}) \rangle = \int d^3\mathbf{R}' T(\mathbf{R}, \mathbf{R}') v_0(\mathbf{R}')$$

Relation between T *and* T^{irr} *operators:*

$$T = T^{irr} \left(1 - G T^{irr}\right)^{-1}$$

Effective viscosity coefficient is given directly by the response operator T^{irr}

Felderhof, Ford and Cohen cluster expansion (1982)

$$T = \sum_{b=1}^{\infty} \sum_{C_1 \ldots C_b} \int dC_1 \ldots dC_b \; n\left(C_1 \ldots C_b\right) S_I(C_1) G \ldots G S_I(C_b)$$

$$T = T^{irr}\left(1 - G T^{irr}\right)^{-1}$$

$$S_I(C_1) \text{———} S_I(C_2) \text{———} S_I(C_3) \text{———} \; \ldots \; \text{———} S_I(C_b)$$

$$n\left(C_1 C_2 C_3 \ldots C_b\right)$$

Diagrammatic approach...

$$S_I(C_1) \text{———} S_I(C_2) \text{———} S_I(C_3) \text{———} \ \ldots \ \text{———} S_I(C_b)$$

$$n\left(C_1 C_2 C_3 \ldots C_b\right)$$

Definition of correlation functions g (between groups of particles):

$$n(C_1) = g(C_1)$$

$$n(C_1 C_2) = g(C_1) g(C_2) + g(C_1|C_2)$$

$$\begin{aligned} n(C_1 C_2 C_3) \quad &= \quad g(C_1) g(C_2) g(C_3) + g(C_1|C_2) g(C_3) \\ &\quad + g(C_1|C_3) g(C_2) + g(C_1) g(C_2|C_3) \\ &\quad + g(C_2|C_2|C_3) \end{aligned}$$

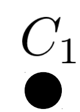
$$S_I(C_1) — S_I(C_2) — S_I(C_3) — \ldots — S_I(C_b)$$

$$n\left(C_1 C_2 C_3 \ldots C_b\right)$$

Diagrammatic representation of correlation functions:

$$n(C_1) = g(C_1)$$

$$n(C_1) =$$



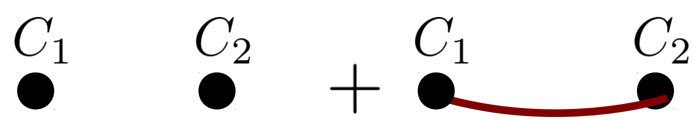
$$S_I(C_1) — S_I(C_2) — S_I(C_3) — \ \ldots \ — S_I(C_b)$$

$$n\left(C_1 C_2 C_3 \ldots C_b\right)$$

Diagrammatic representation of correlation functions:

$$n(C_1 C_2) = g(C_1)g(C_2) + g(C_1|C_2)$$

$$n(C_1 C_2) =$$



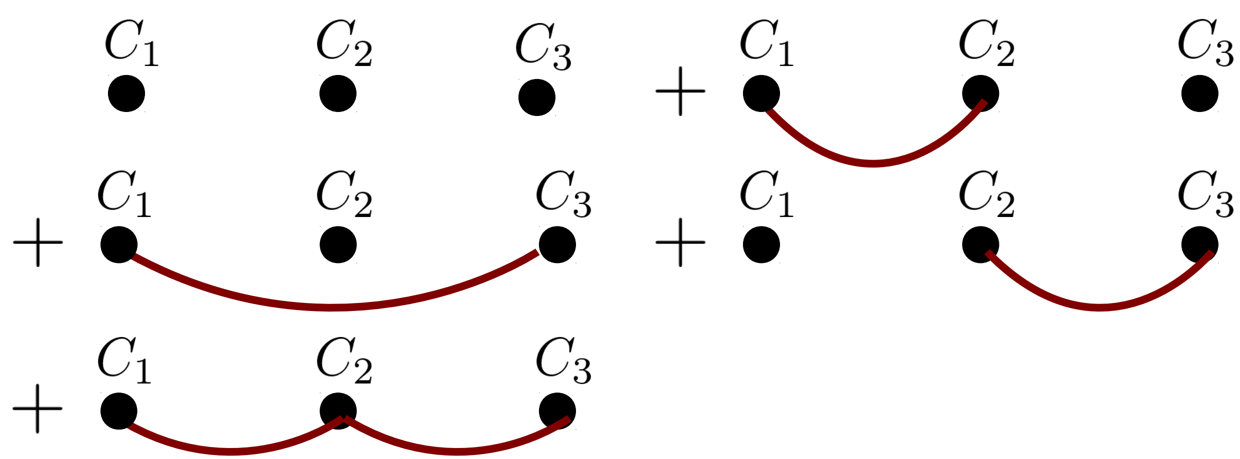
$$S_I(C_1) \text{———} S_I(C_2) \text{———} S_I(C_3) \text{———} \ \dots \ \text{———} S_I(C_b)$$

$$n\left(C_1 C_2 C_3 \dots C_b\right)$$

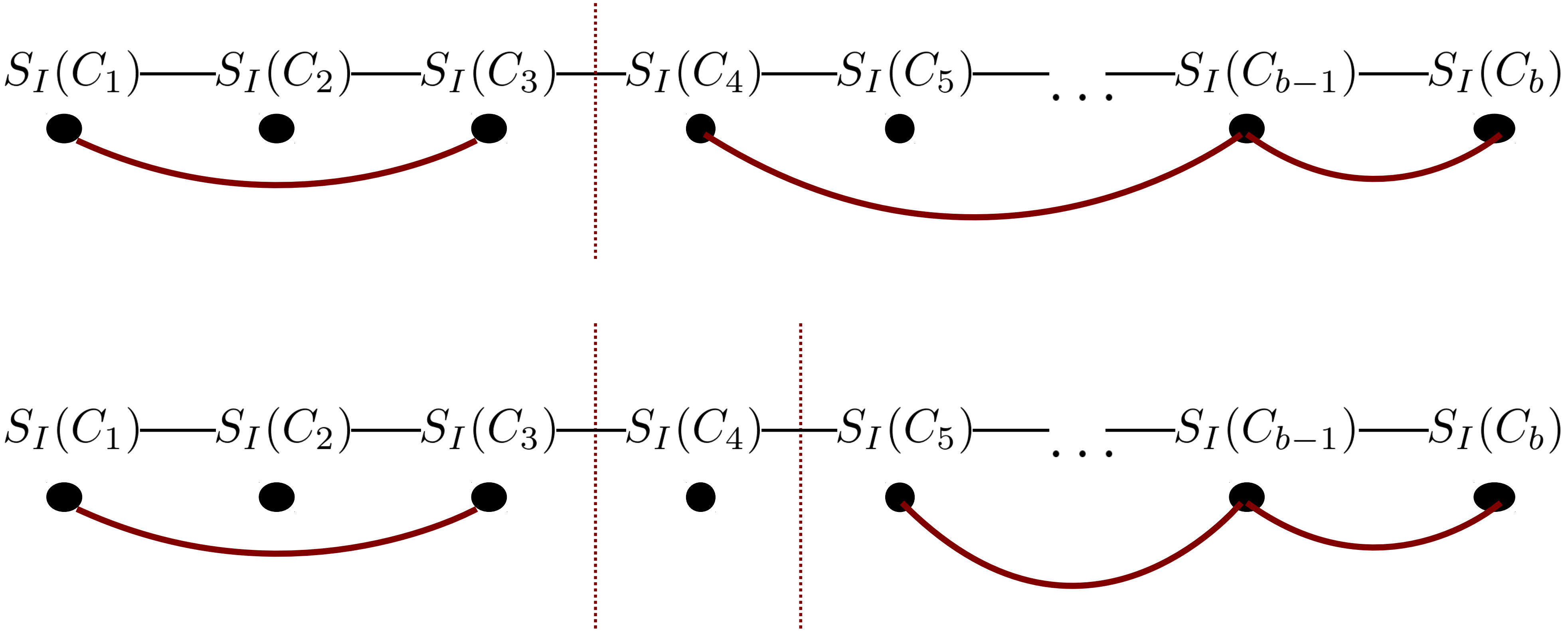
Diagrammatic representation of correlation functions:

$$\begin{aligned} n(C_1 C_2 C_3) \quad &= \quad g(C_1)g(C_2)g(C_3) + g(C_1|C_2)g(C_3) \\ &\quad +g(C_1|C_3)g(C_2) + g(C_1)g(C_2|C_3) \\ &\quad +g(C_1|C_2|C_3) \end{aligned}$$

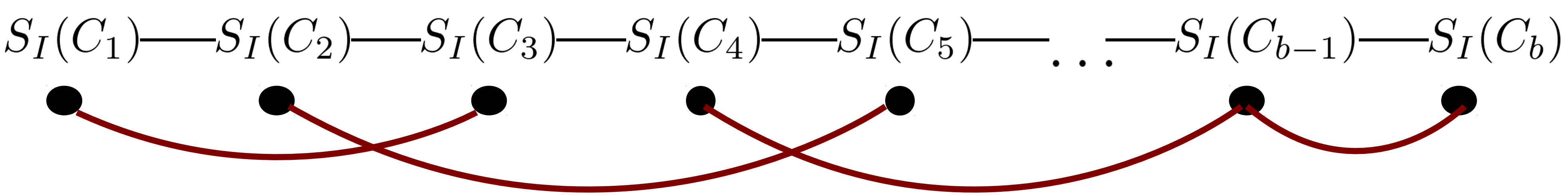
$$n(C_1 C_2 C_3) =$$



Example of reducible diagrams:



Example of irreducible diagram:



$$T = T^{irr} + T^{irr} G T$$

Felderhof, Ford and Cohen cluster expansion (1982)

$$T^{irr} = \sum_{g=1}^{\infty} \sum_{C_1 \ldots C_g} \int dC_1 \ldots dC_g b(C_1| \ldots |C_g) S_I(C_1) G \ldots G S_I(C_g)$$

Block distribution functions (in diagrammatic language):

$b(C_1| \ldots |C_g) =$ *All terms from* $n(C_1 \ldots C_g)$ *giving irreducible diagrams for sequence* $C_1| \ldots |C_g$

Block distribution functions (recurrence formula):

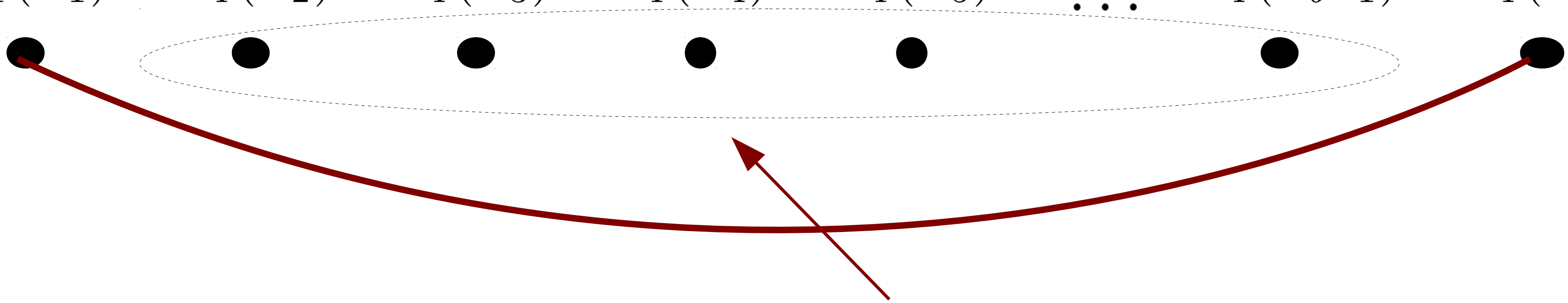
$$b(C) = n(C)$$

$$\begin{aligned} b(C_1| \ldots |C_k|C_{k+1}| \ldots |C_g) &= b(C_1| \ldots |C_k C_{k+1}| \ldots |C_g) \\ &\quad - b(C_1| \ldots |C_k) b(C_{k+1}| \ldots |C_g) \end{aligned}$$

Additional resummation of FFC expansion (2015)

$$T^{irr} = \sum_{g=1}^{\infty} \sum_{C_1 \ldots C_g} \int dC_1 \ldots dC_g b(C_1| \ldots |C_g) S_I(C_1) G \ldots G S_I(C_g)$$

$$S_I(C_1) — S_I(C_2) — S_I(C_3) — S_I(C_4) — S_I(C_5) — \ldots — S_I(C_{b-1}) — S_I(C_b)$$



all correlations are possible here, they yield

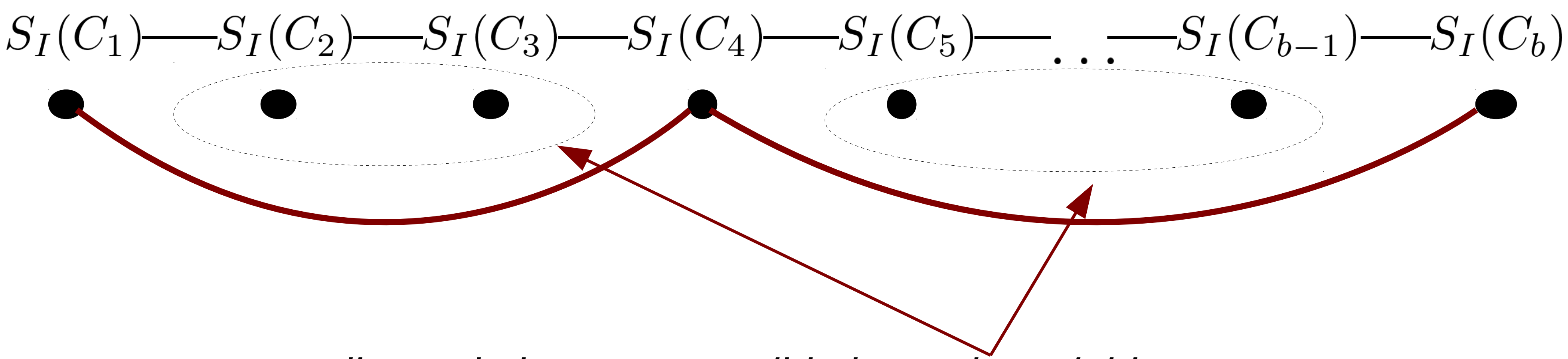
$$n(C_2 \ldots C_{b-1})$$

and after resummation of scattering sequences and integration:

$$G_{eff} = G + GTG$$

Additional resummation of FFC expansion (2015)

$$T^{irr} = \sum_{g=1}^{\infty} \sum_{C_1 \ldots C_g} \int dC_1 \ldots dC_g b(C_1| \ldots |C_g) S_I(C_1) G \ldots G S_I(C_g)$$



all correlations are possible here, they yield

$$n(C_2 C_3) n(C_5 \ldots C_{b-1})$$

and after resummation of scattering sequences and integration:

$$G_{eff} = G + GTG$$

$$T^{irr} = \sum_{b=1}^{\infty} \sum_{C_1 \ldots C_b} \int dC_1 \ldots dC_b H(C_1| \ldots |C_b) S_I(C_1) G_{\text{eff}} \ldots G_{\text{eff}} S_I(C_b)$$

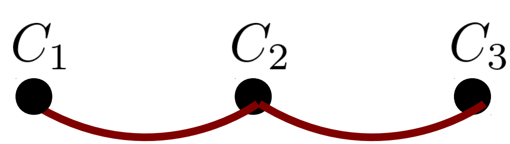
Block correlation functions (in diagrammatic language):

$H(C_1| \ldots |C_g) =$ *All chains from* $n(C_1 \ldots C_g)$

Chains – all terms from $n(C_1 \ldots C_g)$ *which connect all points (also through intersection), e.g.*

$H(C_1) =$ 

$H(C_1|C_2) =$ 

$H(C_1|C_2|C_3) =$ 

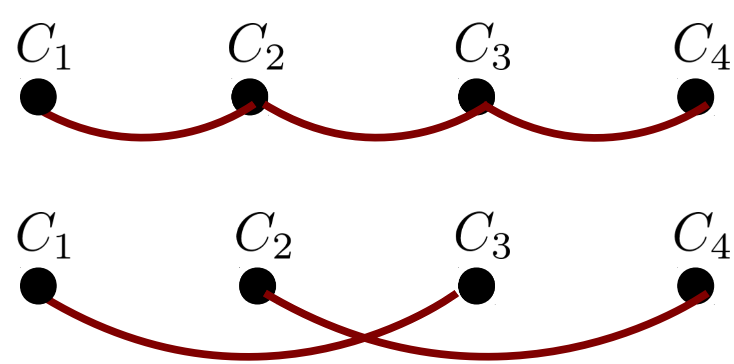
$$T^{irr} = \sum_{b=1}^{\infty} \sum_{C_1 \ldots C_b} \int dC_1 \ldots dC_b H(C_1| \ldots |C_b) S_I(C_1) G_{\text{eff}} \ldots G_{\text{eff}} S_I(C_b)$$

Block correlation functions (in diagrammatic language):

$H(C_1| \ldots |C_g) =$ *All chains from* $n(C_1 \ldots C_g)$

Chains – all terms from $n(C_1 \ldots C_g)$ *which connect all points (also through intersection), e.g.*

$H(C_1|C_2|C_3|C_4) =$ … $+$ …



$$T^{irr} = \sum_{b=1}^{\infty} \sum_{C_1 \dots C_b} \int dC_1 \dots dC_b H(C_1|\dots|C_b) S_I(C_1) G_{\text{eff}} \dots G_{\text{eff}} S_I(C_b)$$

Block correlation functions (in diagrammatic language):

$H(C_1|\dots|C_g) =$ *All chains from* $n(C_1 \dots C_g)$

Block correlation functions (recurrence formula):

$$\mathrm{b}(C_1|\dots|C_b) = \sum_{r=1}^{b-1} \sum_{1=i_1<i_2<\dots<i_{r+1}=b} H(C_{i_1}|\dots|C_{i_{r+1}}) \times n(\{C_{i_1} \dots C_{i_2}\} \setminus \{C_{i_1} C_{i_2}\}) \dots n\left(\{C_{i_r} \dots C_{i_{r+1}}\} \setminus \{C_{i_r} C_{i_{r+1}}\}\right)$$

Comparison of ring expansion with cluster expansion

Felderhof, Ford, Cohen: $$T^{irr} = \sum_{b=1}^{\infty} \sum_{C_1 \ldots C_b} \int dC_1 \ldots dC_b b(C_1| \ldots |C_b) S_I(C_1) G \ldots G S_I(C_b)$$

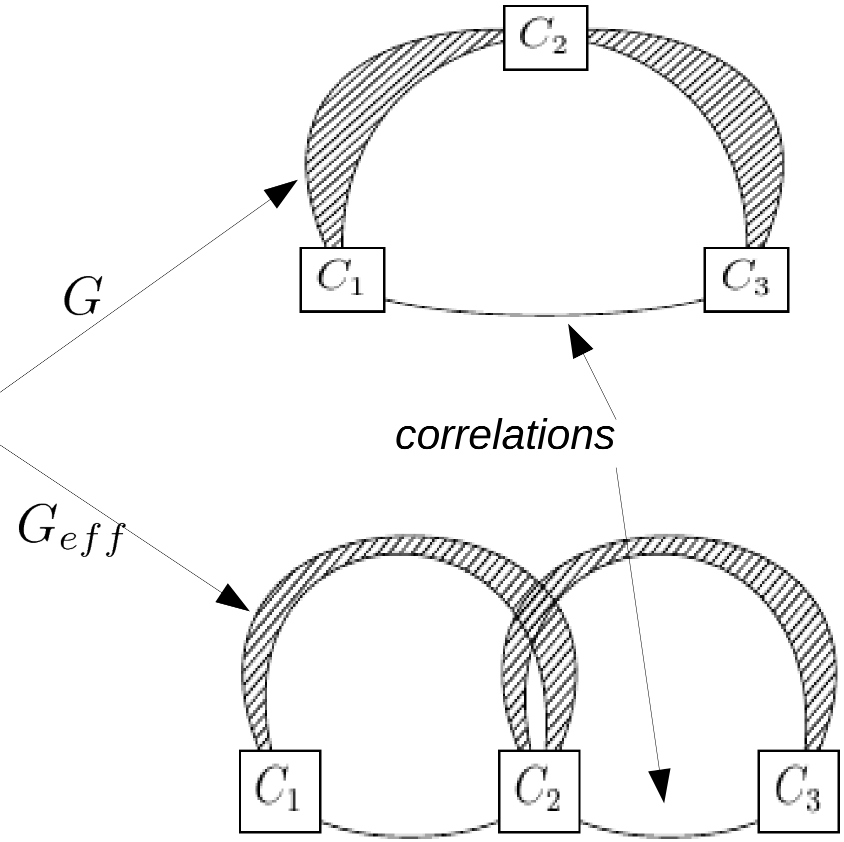
Ring expansion (2015)

$$T^{irr} = \sum_{b=1}^{\infty} \sum_{C_1 \ldots C_b} \int dC_1 \ldots dC_b H(C_1| \ldots |C_b) S_I(C_1) G_{\text{eff}} \ldots G_{\text{eff}} S_I(C_b)$$

When the middle group goes away from others:

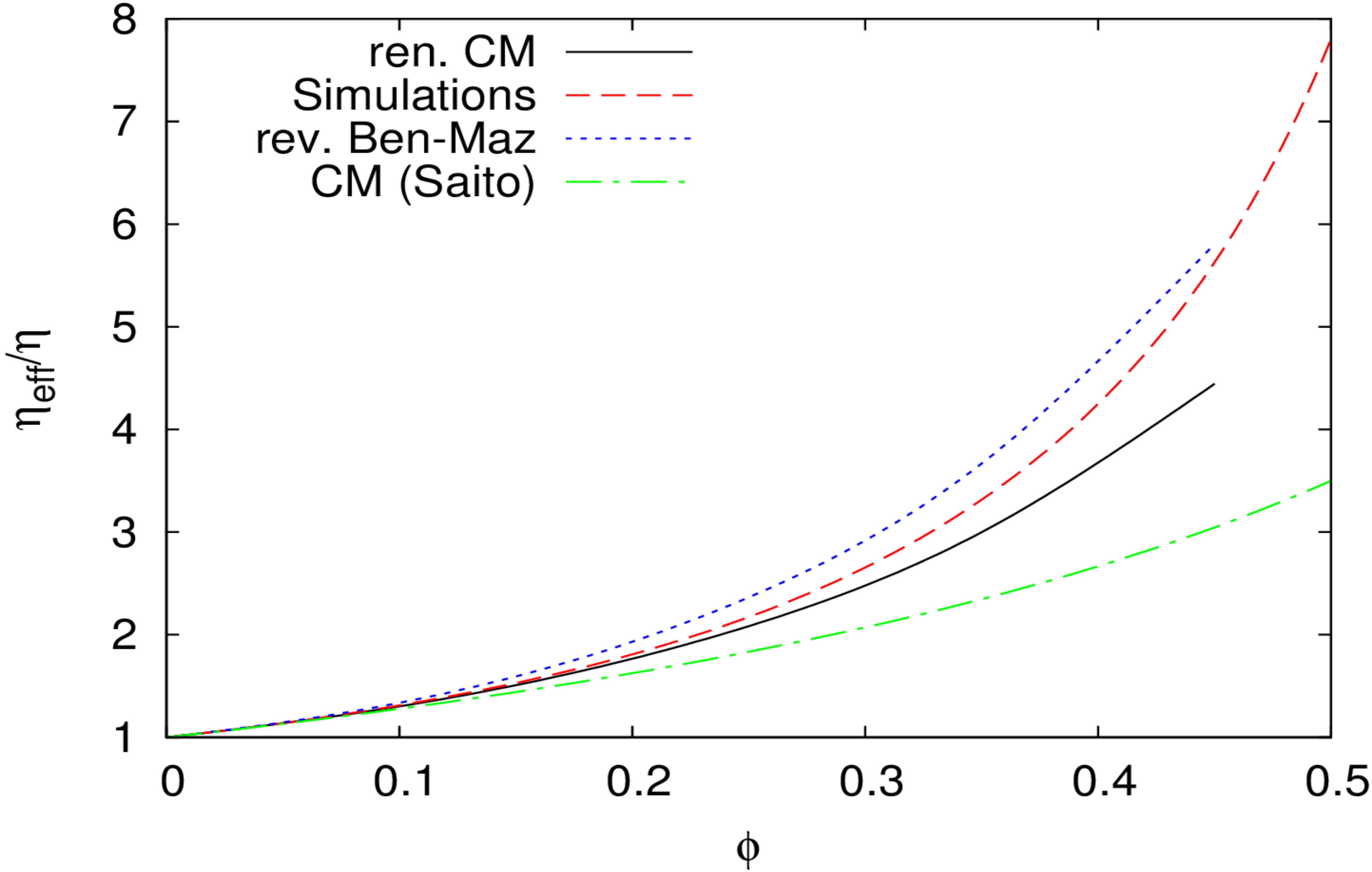
$$b\left(C_1|C_2|C_3\right) \longrightarrow b\left(C_1|C_3\right) b\left(C_2\right)$$

$$H\left(C_1|C_2|C_3\right) \longrightarrow 0$$



Two important differences:
-propagator
-volume of integration

Effective viscosity



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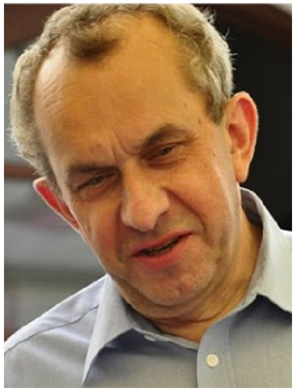
Summary

- Long-range, many-body hydrodynamic interactions and strong interactions of close particles are important in suspensions (still an open problem)
- Rigorous ring expansion can grasp all of the above features (opposite to δγ)
- Generalized Clausius-Mossotti approximation (two-body hydrodynamic interactions not fully taken; comparable to δγ scheme)



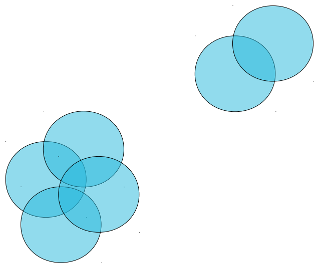
Under suspervision (in years 2005-2011) of Bogdan Cichocki, Univeristy of Warsaw

Outlook

- one-ring approximation (full two-body hydrodynamic interactions, much better accuracy than hitherto theoretical methods in comparison to numerical simulations for volume fraction less than 35%)

- straightforward generalization for different suspensions of spherical particles (droplets, spherical polymers) with different distributions (charged particles)

- polidisperse suspensions



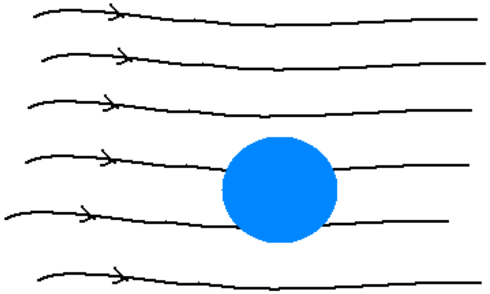
- suspension of nonspherical particles (e.g. double sphere)

Single particle

Lamb (1895) $\mathbf{v}^{+}_{lm\sigma}(\mathbf{r})$

$$l = 1, 2, \ldots, \infty$$
$$m = -l, \ldots, l$$
$$\sigma = 0, 1, 2$$

Single particle in ambient flow $\mathbf{v}_0\left(\mathbf{r}\right)$



$$\mathbf{f}_1\left(\mathbf{r}\right) = \int d\mathbf{r}'\mathbf{M}\left(\mathbf{r}-\mathbf{R}_1, \mathbf{r}'-\mathbf{R}_1\right)\mathbf{v}_0\left(\mathbf{r}'\right)$$

Single particle operator
(Felderhof 1976)

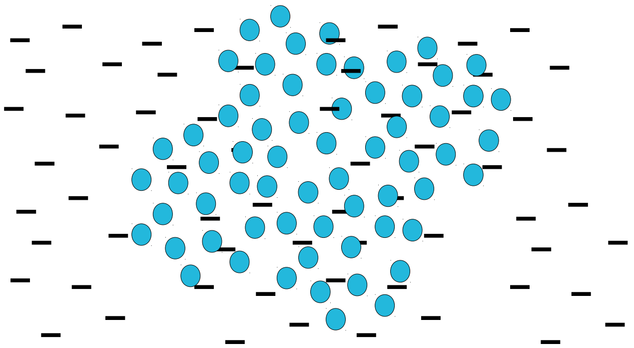
Surface force density
(Cox Brenner (1967); Mazur, Bedeaux (1974))

$$\mathbf{v}\left(\mathbf{r}\right) = \mathbf{v}_0\left(\mathbf{r}\right) + \int d^3\mathbf{r}'\mathbf{G}\left(\mathbf{r}-\mathbf{r}'\right)\cdot\mathbf{f}_1\left(\mathbf{r}'\right)$$

Oseen tensor:

$$\mathbf{G}(\mathbf{r}) = \frac{1}{8\pi\eta}\frac{\mathbf{1}+\hat{\mathbf{r}}\hat{\mathbf{r}}}{r}$$

Suspension



Ambient flow for the particle i in suspension:

$$\mathbf{v}_i(\mathbf{r}) = \mathbf{v}_0(\mathbf{r}) + \sum_{j\neq i} \int d^3\mathbf{r}' \mathbf{G}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{f}_j(\mathbf{r}')$$

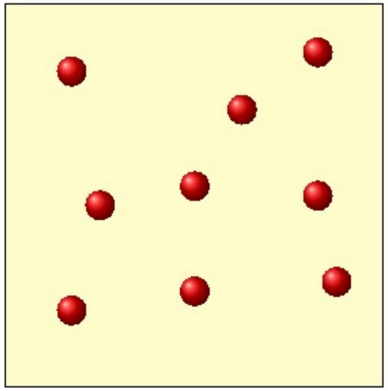
Single particle problem with modified ambient flow

$$\mathbf{f}_i = \mathbf{M}(i) \left(\mathbf{v}_0 + \sum_{i\neq j} \mathbf{G}\mathbf{f}_j \right)$$

Solution in the form of the following scattering series (hydrodynamic interactions)

$$\mathbf{f}_i = \left(\mathbf{M}(i) + \sum_{j\neq i} \mathbf{M}(i)\mathbf{G}\mathbf{M}(j) + \sum_{j\neq i} \sum_{k\neq j} \mathbf{M}(i)\mathbf{G}\mathbf{M}(j)\mathbf{G}\mathbf{M}(k) + \ldots \right) \mathbf{v}_0$$

Scattering series



ambient flow

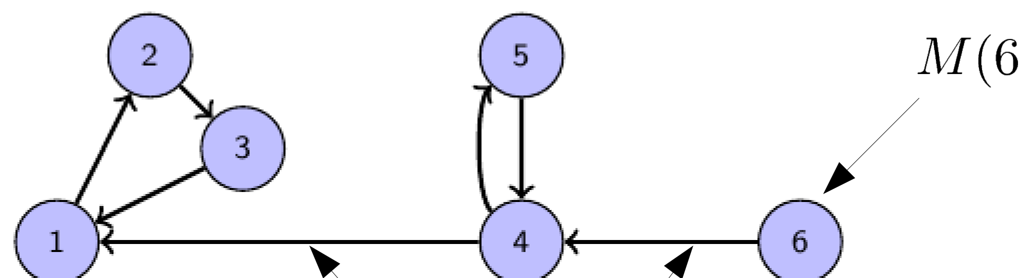
$$\mathbf{f}_i = \left(\mathbf{M}(i) + \sum_{j \neq i} \mathbf{M}(i)\mathbf{G}\mathbf{M}(j) + \sum_{j \neq i} \sum_{k \neq j} \mathbf{M}(i)\mathbf{G}\mathbf{M}(j)\mathbf{G}\mathbf{M}(k) + \ldots \right) \mathbf{v}_0$$

Green function for Stokes equations

Single particle response operator

Example of scattering sequence (many-body):

$$M(1)GM(3)GM(2)GM(1) \times$$
$$G \times M(4)GM(5)GM(4) \times G \times M(6)$$



short range hydrodynamic interactions
(strong interactions of close particles)

long range hydrodynamic interactions (nodal line)

block structure:

$$S_I(C_1) \text{———} S_I(C_2) \text{———} S_I(C_3)$$

$$C_1 \equiv 123 \qquad C_2 \equiv 45 \qquad C_3 \equiv 6$$

Response of suspension

(effective viscosity)

average velocity field of suspension

$$\langle f \rangle (\mathbf{R}) = \int d\mathbf{r}' T^{irr}(\mathbf{R}, \mathbf{R}') \langle v \rangle (\mathbf{R}')$$

average surface dipole force

$$\langle v(\mathbf{R}) \rangle = v_0(\mathbf{R}) + \int d\mathbf{r}' G(\mathbf{R}, \mathbf{R}') \langle f(\mathbf{R}') \rangle$$

$$\langle f(\mathbf{R}) \rangle = \int d^3\mathbf{R}' T(\mathbf{R}, \mathbf{R}') v_0(\mathbf{R}')$$

Relation between T and T^{irr} operators:

$$T = T^{irr} \left(1 - G T^{irr}\right)^{-1}$$

Effective viscosity coefficient is given directly by the response operator T^{irr}

Macroscopic description

Average force density:

$$\langle f(\mathbf{R})\rangle \equiv \left\langle \sum_i f_i \delta(\mathbf{R}-i) \right\rangle$$

Average over probability distribution for configurations of particles, thermodynamic limit

$$\langle f\left(\mathbf{R}\right)\rangle = \int d^3\mathbf{R}' T\left(\mathbf{R},\mathbf{R}'\right) v_0\left(\mathbf{R}'\right)$$

Response operator for suspension in ambient flow

$$T = \sum_{b=1}^{\infty} \sum_{C_1 \ldots C_b} \int dC_1 \ldots dC_b \; n\left(C_1 \ldots C_b\right) S_I(C_1) G \ldots G S_I(C_b)$$

s-particle distribution functions