Stokes law and Einstein viscosity coefficient in complex liquids

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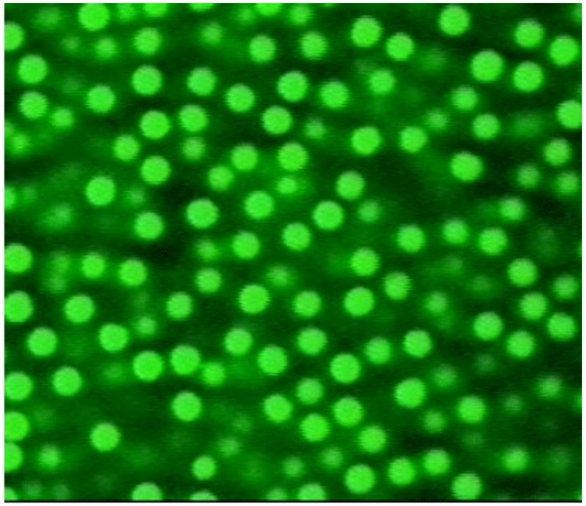
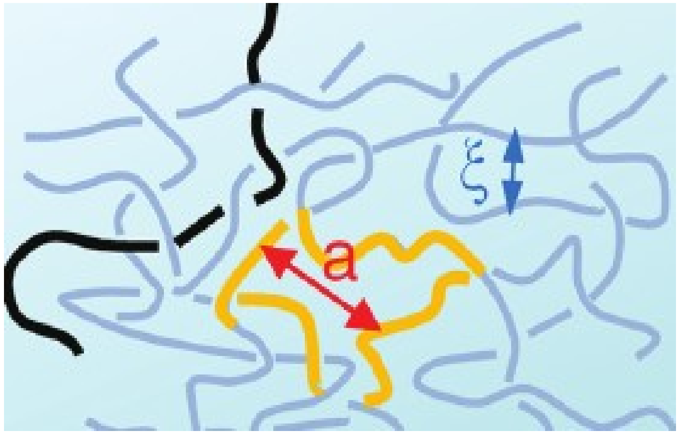
Complex liquids

Liquids with polyatomic structures

Examples: colloidal suspensions, polymer liquids,...





Hydrodynamics of (some) complex liquids

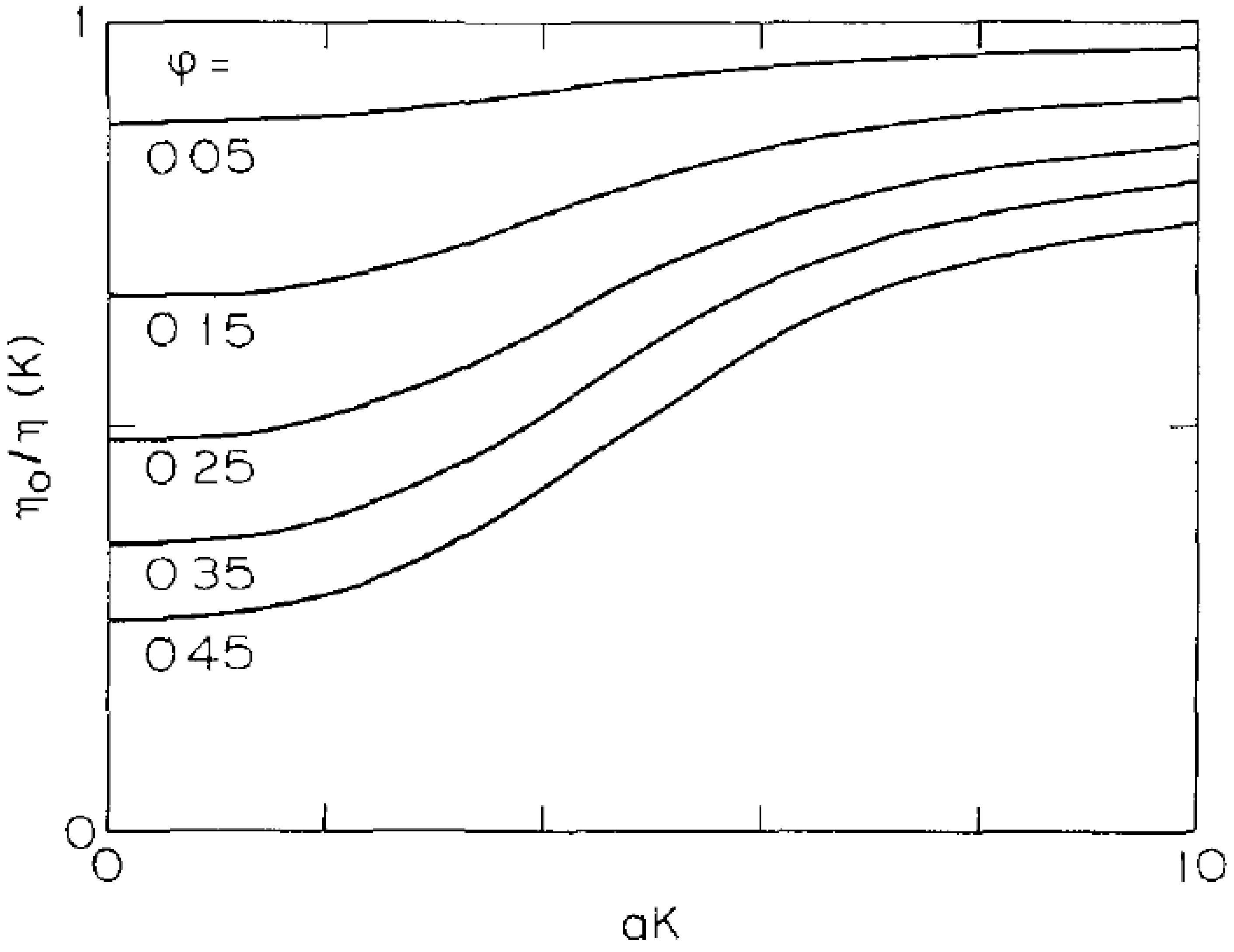
Stokes equations in simple liquids:

$$\begin{aligned} \nabla p(\mathbf{r}) - \eta \Delta \mathbf{v}(\mathbf{r}) &= 0 \\ \nabla \cdot \mathbf{v}(\mathbf{r}) &= 0 \end{aligned}$$

Stokes equations in complex liquids viscosity depends on scale:

$$\begin{aligned} i\mathbf{k}p + k^2 \eta(k)\, \hat{\mathbf{v}}(\mathbf{k}) &= 0, \\ \mathbf{k} \cdot \hat{\mathbf{v}}(\mathbf{k}) &= 0. \end{aligned}$$

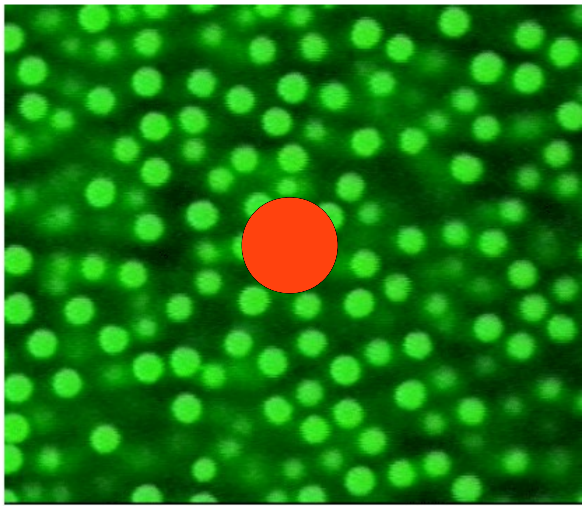
Example of scale-dependent viscosity for monodisperse suspensions

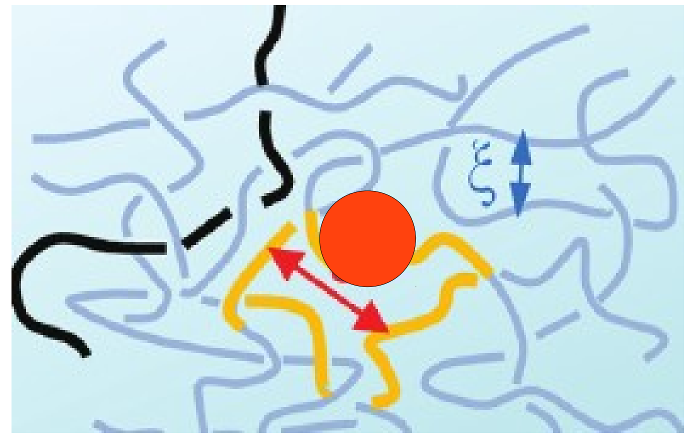


*Beennakker, Physica A, **128**, 48 (1984)*

Goals of our work

Spherical particle immersed in complex liquid





1) Drag force on spherical particle moving in complex liquid:

$$\mathbf{F} = \zeta\left(a\right)\mathbf{U}$$

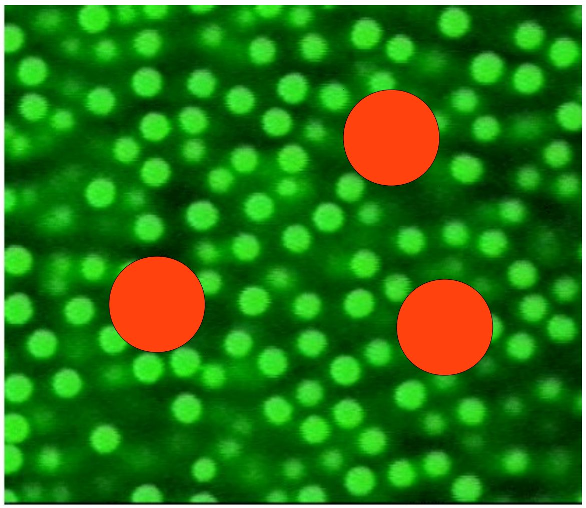
Friction coefficient

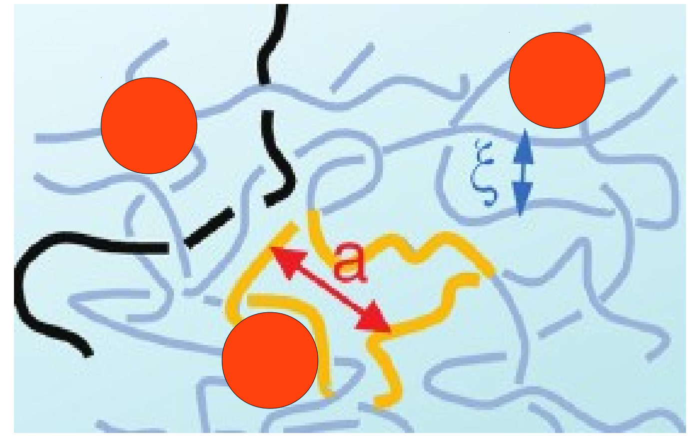
Stokes law in <u>simple liquids</u>:

$$\zeta\left(a\right) = 6\pi\eta a$$

Goals of our work

Add particles to complex liquids





2) Increase of viscosity

$$\eta_{\text{eff}} = \eta\left(0\right)\left(1 + E\left(a\right)\phi\right)$$

Einstein viscosity coefficient

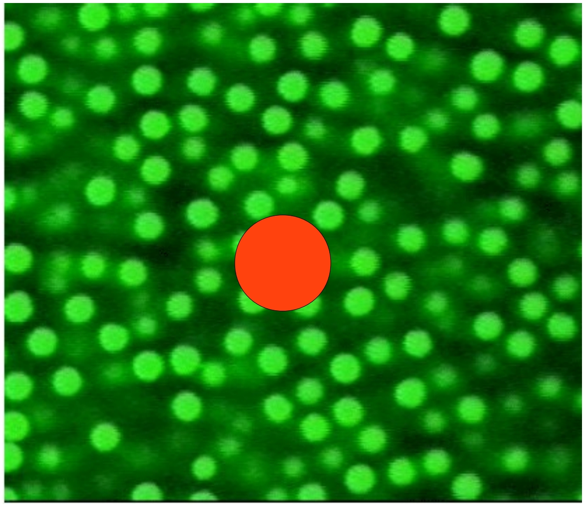
Einstein viscosity coefficient *(simple liquids)*:

$$E\left(a\right) = 5/2$$

Stokes law in complex liquids – formulation of the problem

Stokes equations in complex liquids viscosity depends on scale:

$$
\begin{aligned}
i\mathbf{k}p + k^2\eta(k)\,\hat{\mathbf{v}}(\mathbf{k}) &= 0, \\
\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{k}) &= 0.
\end{aligned}
$$



Boundary conditions:

$\mathbf{v}(\mathbf{r}) = \mathbf{U}$ *for* $|r| = a$

$\mathbf{v}(\mathbf{r}) \to 0$ *for* $r \to \infty$

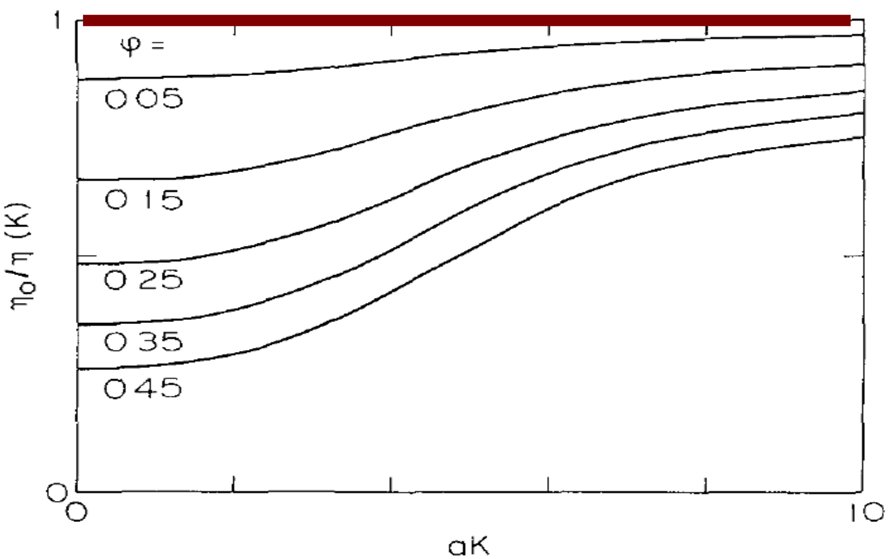
What is the friction coefficient $\zeta(a)$ *?*

$$\mathbf{F} = \zeta(a)\,\mathbf{U}$$

Stokes law in complex liquids – idea of derivation

Difference between simple and complex liquids:

simple liquid		*complex liquid*
$\eta = \mathrm{const}$	→	$\eta(k)$
Second order partial diff. Eqs.		*Infinite order partial diff. Eqs.*



Linearity and spherical symmetry (isotropic fluid, spherical particle) strongly simplifies derivation in simple fluids, $\eta = \mathrm{const}$

Idea: Derive the Stokes law in simple liquids in Fourier space and with use of spherical symmetry, and generalize it to the case of scale dependent viscosity

Stokes law in simple liquids – idea of derivation that uses symmetry and Fourier space

Ansatz for velocity field:

$$\hat{\mathbf{v}}\left(\mathbf{k}\right) = \hat{\mathbf{v}}_0\left(\mathbf{k}\right) + c\hat{\mathbf{v}}_1\left(\mathbf{k}\right)$$

$$\begin{aligned} i\mathbf{k}p_0 + k^2\eta\hat{\mathbf{v}}_0\left(\mathbf{k}\right) &= \mathbf{F}, \\ \mathbf{k}\cdot\hat{\mathbf{v}}_0\left(\mathbf{k}\right) &= 0. \end{aligned}$$

$$\hat{\mathbf{v}}_0\left(\mathbf{k}\right) = \frac{1}{\eta k^2}\left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F}$$

$$\hat{\mathbf{v}}_1\left(\mathbf{k}\right) = \eta k^2\hat{\mathbf{v}}_0\left(\mathbf{k}\right) = \left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F}$$

$$\mathbf{v}_1\left(\mathbf{r}\right) = \frac{1}{4\pi r^3}\left(\mathbf{1} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\mathbf{F}$$

Boundary conditions on the surface of particle, $\mathbf{v}\left(a\hat{\mathbf{r}}\right) = \mathbf{U}$
applied to the above ansatz lead to $c, \mathbf{F}$

$$\mathbf{F} = 6\pi\eta a\mathbf{U}$$

Stokes law – generalization to the case of complex liquids

Ansatz for velocity field:

$$\hat{\mathbf{v}}(\mathbf{k}) = \hat{\mathbf{v}}_0(\mathbf{k}) + c\hat{\mathbf{v}}_1(\mathbf{k})$$

$$\begin{aligned} i\mathbf{k}p_0 + k^2\eta(k)\,\hat{\mathbf{v}}_0(\mathbf{k}) &= \mathbf{F}, \\ \mathbf{k}\cdot\hat{\mathbf{v}}_0(\mathbf{k}) &= 0. \end{aligned}$$

$$\hat{\mathbf{v}}_0(\mathbf{k}) = \frac{1}{\eta(k)\,k^2}\left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F}$$

$$\hat{\mathbf{v}}_1(\mathbf{k}) = \eta(k)\,k^2\hat{\mathbf{v}}_0(\mathbf{k}) = \left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F} \qquad \mathbf{v}_1(\mathbf{r}) = \frac{1}{4\pi r^3}\left(\mathbf{1} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\mathbf{F}$$

Boundary conditions on the surface of particle, $\mathbf{v}(a\hat{\mathbf{r}}) = \mathbf{U}$
applied to the above ansatz lead to

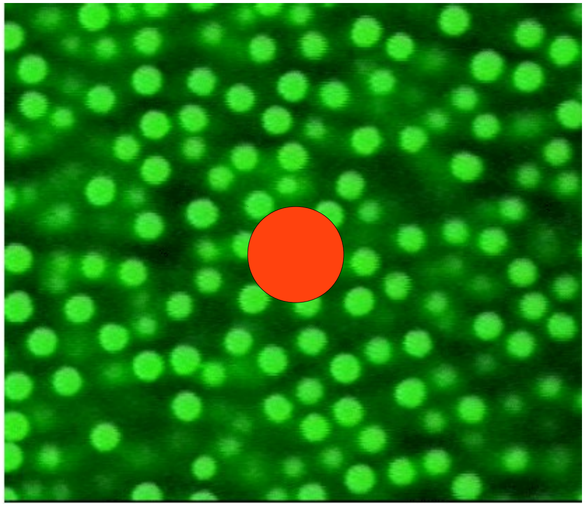
$$\mathbf{U} = \frac{1}{(2\pi)^3}\int d^3k\; e^{ia\hat{\mathbf{r}}\mathbf{k}}\frac{1}{\eta(k)\,k^2}\left(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}\right)\mathbf{F} + c\frac{1}{4\pi r^3}\left(\mathbf{1} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\mathbf{F}$$

which yields...

Stokes law in complex liquids

What is the friction coefficient?

$$\mathbf{F} = \zeta\left(a\right)\mathbf{U}$$



$$\zeta\left(a\right) = 3\pi^2 / \left(\int_0^\infty dk\, j_0\left(ka\right)/\eta\left(k\right)\right)$$

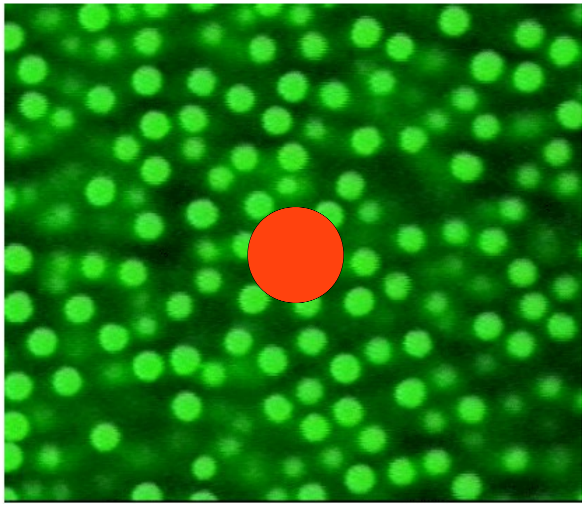
$$j_0\left(x\right) = \sin\left(x\right)/x$$

Application of the Stokes law in complex liquids
- from friction coefficient to scale dependent viscosity

Stokes law: from scale dependent viscosity to friction coefficient

$$\zeta(a) = 3\pi^2 / \left(\int_0^\infty dk \, j_0(ka) / \eta(k) \right)$$

Fourier transform

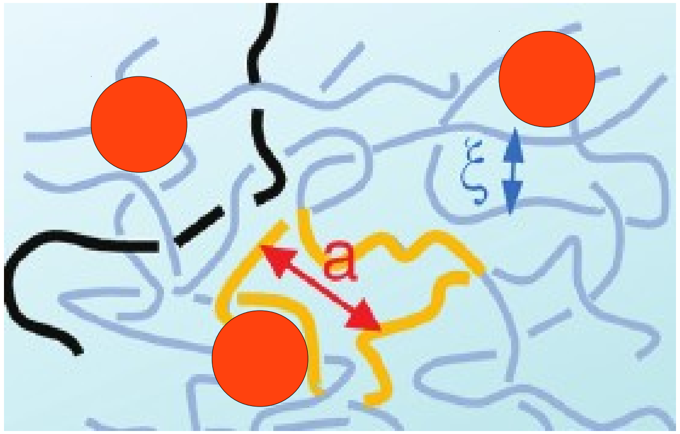


friction coefficient from scale dependent viscosity:

$$\eta(k) = \frac{1}{6\pi k^2} \left[\int_0^\infty da \, a^2 \frac{j_0(ak)}{\zeta(a)} \right]^{-1}$$

Einstein viscosity coefficient E(a) in complex liquids

$$\eta_{\text{eff}} = \eta(0)\left(1 + E(a)\,\phi\right)$$



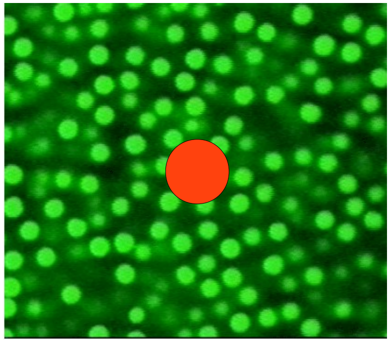
$$E(a) = -\frac{5}{2}\frac{1}{\eta(0)}\left[\frac{10}{3\pi}a^2\frac{d}{da}\int_0^\infty dk\,\frac{j_0(ak)}{\eta(k)} + \frac{4}{\pi}a\int_0^\infty dk\,\frac{j_2(ak)}{\eta(k)} + \frac{4a^2}{3\pi}\frac{d}{da}\int_0^\infty dk\,\frac{j_2(ak)}{\eta(k)}\right]^{-1}$$

$$j_0(x) = \sin(x)/x$$

$$j_2(x) = -\sin(x)/x - 3\cos x/x^2 + 3\sin(x)/x^3$$

Einstein viscosity coefficient and Stokes friction coefficient

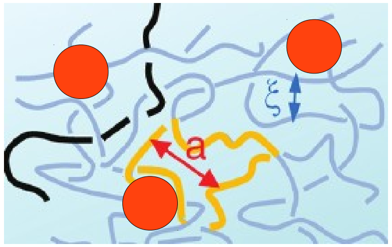
Stokes law in complex liquids:



$$\zeta\left(a\right) = 3\pi^2 / \left(\int_0^\infty dk\, j_0\left(ka\right)/\eta\left(k\right) \right)$$

Einstein viscosity coefficient in complex liquids:

$$E\left(a\right) = -\frac{5}{2}\frac{1}{\eta\left(0\right)}\left[\frac{10}{3\pi}a^2\frac{d}{da}\int_0^\infty dk\,\frac{j_0\left(ak\right)}{\eta\left(k\right)} + \frac{4}{\pi}a\int_0^\infty dk\,\frac{j_2\left(ak\right)}{\eta\left(k\right)} + \frac{4a^2}{3\pi}\frac{d}{da}\int_0^\infty dk\,\frac{j_2\left(ak\right)}{\eta\left(k\right)}\right]^{-1}$$



Relation between friction and Einstein viscosity coefficient

$$E\left(a\right) = \frac{5}{12\pi\eta\left(0\right)a^2}\frac{\zeta\left(a\right)^2}{\frac{d\zeta(a)}{da}}$$

Summary

We derive:

- Stokes law in complex liquids,
- Einstein viscosity coefficient,

  when viscosity is scale-dependent (depends on wave vector)

In collaboration with

Piotr Garstecki

Robert Hołyst